Elastic and dynamic form factors of an atomic nucleus in the shell model with correction for the center-of-mass motion[1]

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Abstract

Analytical expressions for the elastic and dynamic form factors (FFs) are derived in the shell model (SM) with a potential well of finite depth. The consideration takes into account the motion of the target-nucleus center of mass (CM). Explanation is suggested for a simultaneous shrinking of the density and momentum distributions of nucleons in nuclei. The convenient working formulae are given to handle the expectation values of relevant multiplicative operators in case of the $1s-1p$ shell nuclei.

1. It is known that nuclear SM wave functions (WFs) do not possess the property of translational invariance (TI). Several methods in earlier and recent studies of nuclear structure (see, e.g., $[1-4]$) have been proposed to transform any WF into one which is translationally invariant. Of great interest among these methods is the projection procedure considered in [4]. Along with other attractive features shown in [4], this procedure enables a comparatively simple evaluation of the corresponding CM correction to the purely shell quantities (see Ref. [5]).

The approach developed in [5] is extended here to calculate the cross sections of the elastic and quasifree electron scattering on atomic nuclei with single-particle (s.p.) configurations more complex than the $1s^4$ one. In particular, we pay special attention to the physical interpretation of a simultaneous shrinking of the density and momentum distributions of nucleons in a nucleus due to the employed separation of its CM motion.

2. By definition, the elastic FF in question is

$$F(\vec{q}) = \langle \Phi_{intr} \mid \exp[\imath\vec{q}(\hat{\vec{r}}_1 - \hat{\vec{R}})] \mid \Phi_{intr} \rangle, \tag{1}$$

[1]Paper published in Ukr.J.Phys. **22** (1977) 1646, extended and translated from Ukrainian into English by authors
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while the dynamic FF can be written as in [5],

$$S(\vec{q},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\imath a\tau) S(\vec{q},\tau) d\tau, \tag{2}$$

$$S(\vec{q},\tau) = \langle \Phi_{intr} \mid \exp[\imath(\hat{\vec{p}}_1 - A^{-1}\hat{\vec{P}})\vec{q}m^{-1}\tau] \mid \Phi_{intr}\rangle,$$

$$a = \omega + q^2/2m,$$

where $\omega(\vec{q})$ is the energy (momentum) transfer, m the nucleon mass,

$$\hat{\vec{R}} = A^{-1}\sum_{\alpha=1}^{A} \hat{\vec{r}}_\alpha \qquad (\hat{\vec{P}} = \sum_{\alpha=1}^{A} \hat{\vec{p}}_\alpha)$$

the CM coordinate (total momentum) operator of the nucleus composed of A nucleons so that $\hat{\vec{r}}_\alpha(\hat{\vec{p}}_\alpha)$ the coordinate (momentum) operator for nucleon number α, and Φ_{intr} the intrinsic WF of the nuclear ground state (g.s.).

Following [4], we take as $\mid \Phi_{intr}\rangle$ the vector

$$\mid \Phi_{intr}\rangle = (\vec{R} = 0 \mid \Phi\rangle[\langle\Phi \mid \vec{R} = 0)(\vec{R} = 0 \mid \Phi\rangle]^{-1/2}, \tag{3}$$

for a given trial (approximate) WF Φ that may be nontranslationally invariant (nTI). Here, a round bracket, $|\,)$, is used to represent a vector in the space of the CM coordinate only.

In the harmonic oscillator (HO) model, where the Slater determinant $\mid \Phi\rangle$ is "pure" in the space of the CM coordinate (the Bethe-Rose-Elliot-Skyrme theorem [6, 7]), one has

$$F(q) = \exp(\frac{q^2 r_0^2}{4A}) F_0(q) \tag{4}$$

with

$$F_0(q) = \langle \Phi \mid \exp(\imath\vec{q}\hat{\vec{r}}_1) \mid \Phi\rangle \tag{4'}$$

and

$$S(\vec{q},\tau) = \exp(\frac{b^2\tau^2}{4A}) S_0(\vec{q},\tau) \tag{5}$$

with

$$S_0(\vec{q},\tau) = \langle \Phi \mid \exp(\imath\hat{\vec{p}}_1\vec{q}\tau/m) \mid \Phi\rangle \tag{5'}$$

Here r_0 is the oscillator parameter, $b = p_0 q/m, \quad p_0 = r_0^{-1}$.

Result (4) is widely used in applications starting from the work [8]. Note also that Eqs.(4) and (5) are valid within the HO model, being independent of any specific way to separate the CM

motion (e.g., the Ernst-Shakin-Thaler prescription (3) that is equivalent to the so-called "fixed CM approximation" in case of finite nuclei (bound systems)).

Now, before finding some analogs of relations (4) – (5) with an arbitrary WF Φ (in particular, the Slater determinant constructed of the s.p. orbitals in a potential well of finite depth, say, the Woods-Saxon or Hartree-Fock ones), we would like to trace the CM corrections of the density and momentum distributions $\rho(r)$ and $\eta(p)$ within the HO model. In this connection, let us recall the general definitions for these quantities of primary concern:

$$\rho(\vec{r}) \equiv \langle \Phi_{intr} \mid \delta(\hat{\vec{r}}_1 - \hat{\vec{R}} - \vec{r}) \mid \Phi_{intr} \rangle = (2\pi)^{-3} \int \exp(-\imath\vec{q}\vec{r}) F(\vec{q}) d\vec{q} \tag{6a}$$

and

$$\begin{aligned}\eta(\vec{p}) &\equiv \langle \Phi_{intr} \mid \delta(\hat{\vec{p}}_1 - \hat{\vec{P}}/A - \vec{p}) \mid \Phi_{intr} \rangle = \\ &= (2\pi)^{-3} (\tau/m)^3 \int \exp(-\imath\vec{p}\vec{q}\tau/m) S(\vec{q},\tau) d\vec{q}\end{aligned} \tag{6b}$$

For the $1s$-$1p$ shell nuclei we find in the HO model,

$$F_0(\vec{q}) = (1 - \frac{A-4}{6A} q^2 r_0^2) \exp(-\frac{1}{4} q^2 r_0^2),$$

$$S_0(\vec{q},\tau) = (1 - \frac{A-4}{6A} b^2 \tau^2) \exp(-\frac{1}{4} b^2 \tau^2)$$

Substituting these expressions, respectively, into Eq.(4) and Eq. (5) we get with the help of formulae (6):

$$\rho(r) = \pi^{-3/2} \bar{r}_0^{-3} (\frac{3}{A-1} + \frac{2}{3} \frac{A-4}{A-1} \frac{r^2}{\bar{r}_0^2}) \exp(-\frac{r^2}{\bar{r}_0^2}), \tag{7a}$$

$$\eta(p) = \pi^{-3/2} \bar{p}_0^{-3} (\frac{3}{A-1} + \frac{2}{3} \frac{A-4}{A-1} \frac{p^2}{\bar{p}_0^2}) \exp(-\frac{p^2}{\bar{p}_0^2}), \tag{7b}$$

$$\bar{r}_0 = \sqrt{\frac{A-1}{A}} r_0, \quad \bar{p}_0 = \sqrt{\frac{A-1}{A}} p_0$$

The intrinsic distributions without any CM correction are

$$\rho_0(r) \equiv \langle \Phi \mid \delta(\hat{\vec{r}}_1 - \vec{r}) \mid \Phi \rangle = \pi^{-3/2} r_0^{-3} (\frac{4}{A} + \frac{2(A-4) r^2}{3A r_0^2}) \exp(-\frac{r^2}{r_0^2}), \tag{8a}$$

$$\eta_0(p) \equiv \langle \Phi \mid \delta(\hat{\vec{p}}_1 - \vec{p}) \mid \Phi \rangle = \pi^{-3/2} p_0^{-3} (\frac{4}{A} + \frac{2(A-4) p^2}{3A p_0^2}) \exp(-\frac{p^2}{p_0^2}), \tag{8b}$$

By comparing Eqs.(7) and (8) one can see that the density and momentum distributions are subject to the equal changes. It does not seem to be surprising if we invoke the well known symmetry between the coordinate and momentum representations of the HO model Hamiltonian. One should

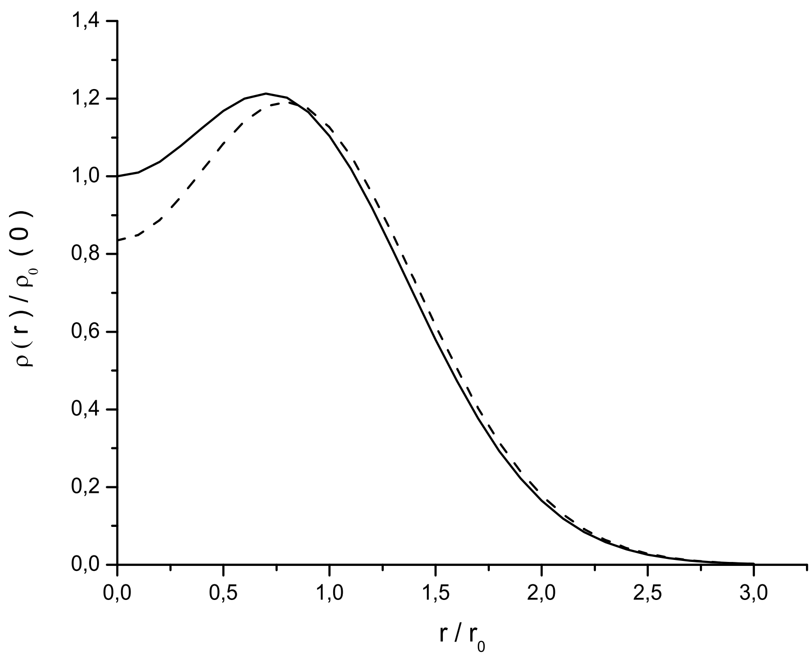

Figure 1: The variation of $\rho_0(r)/\rho_0(0)$ (solid) and of $\rho(r)/\rho_0(0)$ (dashed) with r/r_0

point out that, apart from the "most symmetrical" nucleus with $A = 4$, these CM corrections are not reduced to a simple renormalization of the oscillator parameter (e.g., the change $r_0 \rightarrow \bar{r}_0$).

The dependences $\rho(r)$ and $\rho(r)_0$ calculated by formulae (7) and (8) with $A = 16$ are depicted in Fig.1 In this context, we show the ratio

$$\frac{\rho_0(r)}{\rho_0(0)} = [1 + \frac{1}{6}(A-4)\frac{r^2}{r_0^2}] \exp(-\frac{r^2}{r_0^2})$$

versus the ratio

$$\frac{\rho(r)}{\rho_0(r)} = \frac{3}{4}[\frac{A}{A-1}]^{5/2}[1 + \frac{2}{9}(A-4)\frac{r^2}{\bar{r}_0^2}] \exp(-\frac{r^2}{\bar{r}_0^2})$$

We see that an additional correlation between the nucleons, incorporated into (inherent in) the intrinsic density $\rho(r)$, gives rise to their redistribution between the nuclear shells (from the $1s$-shell to the $1p$-one, while the center of the $1p$-distribution is shifted toward larger r-values). This rearrangement of the nuclear interior is accompanied by a decrease of the nuclear density in its peripheral region. It implies the corresponding increase of the probability to find the nucleons in the central part of the nucleus. Remind that both the density distributions (DDs) $\rho(\vec{r})$ and $\rho_0(\vec{r})$ are normalized to unity. We interpret these features of the intrinsic DD $\rho(r)$ as its shrinking

in comparison with the DD $\rho_0(r)$, which embodies the spurious CM motion modes. The same interpretation can be applied to the momentum distribution (MD) $\eta(p)$ vs $\eta_0(p)$.

Such a simultaneous shrinking of the DD and MD becomes more tractable if one evaluate the respective r.m.s. radii and momenta. By definition, one has

$$\langle r^2\rangle \equiv \int r^2\rho(\vec{r})d\vec{r}, \quad \langle p^2\rangle \equiv \int p^2\eta(\vec{p})d\vec{p} \tag{9}$$

It is readily seen that

$$\langle r^2\rangle_0 \equiv \int r^2\rho_0(\vec{r})d\vec{r} = \langle r^2\rangle + (000 \mid \hat{\vec{R}}^2 \mid 000), \tag{10a}$$

$$\langle p^2\rangle_0 \equiv \int p^2\eta_0(\vec{p})d\vec{p} = \langle p^2\rangle + A^{-2}(000 \mid \hat{\vec{P}}^2 \mid 000), \tag{10b}$$

where the vector $\mid 000)$ is used to describe the "zero" (lowest-energy) CM oscillations with respect to the origin of coordinates. A complementary smearing of $\rho_0(r)$ and $\eta_0(p)$ compared respectively with $\rho(r)$ and $\eta(p)$ is due to the nonphysical motion mode.

Further, let us consider the commutation relations for the relative coordinates $\hat{\vec{r}}\,'_\alpha = \hat{\vec{r}}_\alpha - \hat{\vec{R}}$ and the canonically conjugate momenta $\hat{\vec{p}}\,'_\alpha = \hat{\vec{p}}_\alpha - \hat{\vec{P}}/A$ $(\alpha = 1, \ldots, A)$:

$$[(\hat{\vec{r}}\,'_\alpha)_k, (\hat{\vec{p}}\,'_\alpha)_l] = \imath\delta_{kl}(1 - A^{-1}) \ (k, l = 1, 2, 3) \tag{11}$$

along with the original ones:

$$[(\hat{\vec{r}}_\alpha)_k, (\hat{\vec{p}}_\alpha)_l] = \imath\delta_{kl} \ (k, l = 1, 2, 3) \tag{11'}$$

The corresponding uncertainties meet the equations (see, for instance, [9], p. 54, and also Suppl. A to this translation) :

$$\langle\Phi \mid (\widehat{\Delta\vec{r}})^2 \mid \Phi\rangle\langle\Phi \mid (\widehat{\Delta\vec{p}})^2 \mid \Phi\rangle = \langle r^2\rangle_0\langle p^2\rangle_0 \geq \frac{9}{4}, \tag{12}$$

$$\langle\Phi \mid (\widehat{\Delta\vec{r}}\,')^2 \mid \Phi\rangle\langle\Phi \mid (\widehat{\Delta\vec{r}}\,')^2 \mid \Phi\rangle = \langle r^2\rangle_\Phi\langle p^2\rangle_\Phi \geq \frac{9}{4}\left(\frac{A-1}{A}\right)^2 \tag{12'}$$

for any state $\mid \Phi\rangle$ normalized to unity, where we have introduced the expectation values $\langle r^2\rangle_\Phi = \langle\Phi \mid (\hat{\vec{r}}_1')^2 \mid \Phi\rangle$ and $\langle p^2\rangle_\Phi = \langle\Phi \mid (\hat{\vec{p}}_1')^2 \mid \Phi\rangle$. The latter are converted into the values $\langle r^2\rangle$ and $\langle p^2\rangle$ if $\mid \Phi\rangle = \mid \Phi_{intr}\rangle$.

Alternatively, according Eqs.(10) we find in the HO model that

$$\langle r^2\rangle\langle p^2\rangle = \langle r^2\rangle_0\langle p^2\rangle_0 \left[1 - \frac{(000 \mid \hat{\vec{R}}^2 \mid 000)}{\langle r^2\rangle_0}\right]\left[1 - \frac{(000 \mid \hat{\vec{P}}^2 \mid 000)}{A^2\langle p^2\rangle_0}\right] \tag{13}$$

Then, taking into account that

$$(000 \mid \hat{\vec{R}}^2 \mid 000) = \frac{3}{2A} r_0^2, \quad (000 \mid \hat{\vec{P}}^2 \mid 000) = \frac{3A}{2} p_0^2$$

and that in the HO model for a nucleus with fully occupied (closed) shells

$$\langle r^2 \rangle_0 = \varkappa r_0^2, \;\; \langle p^2 \rangle_0 = \varkappa p_0^2,$$

$$\varkappa = 2 \sum_{N=0}^{N_{max}} (N+1)(N+2)(N+3/2)/A, \quad A = 2 \sum_{N=0}^{N_{max}} (N+1)(N+2), \tag{14}$$

i.e.,

$$\varkappa = \frac{3}{2} + \sum_N N(N+1)(N+2) / \sum_N (N+1)(N+2),$$

where N is the principal quantum number, we get

$$\langle r^2 \rangle_0 \langle p^2 \rangle_0 = \varkappa^2 \geq \frac{9}{4} \tag{15}$$

$$\langle r^2 \rangle \langle p^2 \rangle = \varkappa^2 \left(\frac{A-1}{A} \right)^2 \left[1 + \frac{1}{A-1} \frac{\kappa - 3/2}{\kappa} \right]^2 \geq \frac{9}{4} \left(\frac{A-1}{A} \right)^2 \tag{15'}$$

Thus we arrive again to relations (12) and (12′). From this derivation it follows that Eqs. (15) and (15′) without the signs of inequality are permitted only for the $1s^4$ configuration in the HO model. Such a minimization of the uncertainty relations is retained after making the CM correction.

3. Further, using the algebraic procedure applied in [5] for the calculation of the expectation values (1) and (2) with the Slater determinant $|\Phi\rangle = |(1s)^4\rangle$ of the simple HO orbitals, we find

$$F(\vec{q}) = \exp(-\frac{q^2 \bar{r_0}^2}{4}) U(\vec{q})/U(0), \tag{16}$$

$$U(\vec{q}) = \int \exp(-\frac{\lambda^2 r_0^2}{4A}) f(\vec{\lambda}, \vec{q}) d\vec{\lambda},$$

$$f(\vec{\lambda}, \vec{q}) = \langle \Phi \mid \hat{O}_1(\vec{b}) \prod_{\alpha=2}^{A} \hat{O}_\alpha(\vec{c}) \mid \Phi \rangle$$

with the operators

$$\hat{O}_\alpha(\vec{x}) = \exp(-\vec{x}^* \; \hat{\vec{a}}_\alpha^\dagger) \exp(-\vec{x} \; \hat{\vec{a}}_\alpha), \;\; (\alpha = 1, \ldots, A)$$

where $\vec{b} = \vec{c} + \imath \frac{r_0}{\sqrt{2}} \vec{q}$, $\vec{c} = \imath \frac{r_0}{\sqrt{2A}} (\vec{\lambda} - \vec{q})$, $\hat{\vec{a}}^\dagger (\hat{\vec{a}})$ is the vector whose components are the creation (annihilation) operators for oscillator quanta in the three different space directions.

Analogously, one can show that

$$S(\vec{q}, \tau) = \exp(-\frac{q^2 \bar{p_0}^2}{4m} \tau^2) U(-\imath p_0^2 \frac{\vec{q}}{m} \tau)/U(0) \tag{17}$$

Now, let us assume that the many-body state $\mid \Phi\rangle$ is a Slater determinant of s.p. states $\mid \phi_\gamma\rangle$ $(\gamma = 1, \ldots, A)$ which describe completely occupied bound states of the nucleons in a spherically symmetric potential well (e.g., Woods-Saxon potential or Hartree-Fock field). Then (see Suppl. B)

$$f(\vec{\lambda}, \vec{q}) = A^{-1} \sum_{\rho=1}^{A} D_\rho, \tag{18}$$

where D_ρ is the determinant that is deduced from the determinant

$$D = \| \langle \phi_\gamma \mid \hat{O}(\vec{c}) \mid \phi_{\gamma'}\rangle \|$$

replacing the vector $\vec{c}$ by $\vec{b}$ in its column with the label ρ.

As an illustration, let us consider the $(1s)^4(1p)^{12}$ configuration in the ls-coupling scheme. From Eq.(18) it follows that (see Suppl. C)

$$f(\vec{\lambda}, \vec{q}) = \frac{1}{4} d^3 (d_1 + d_2 + d_3 + d_4), \tag{19}$$

where $d = \| \langle \phi_{nlm} \mid \hat{O}(\vec{c}) \mid \phi_{n'l'm'}\rangle \|$ is the 4×4 determinant, $\mid \phi_{nlm}\rangle$ the spatial part of the vector $\mid \phi_\gamma\rangle$ in the shell with the radial quantum number n, the orbital angular momentum l and its projection m, and $d_i (i = 1, \ldots, 4)$ are deduced from d just as D_ρ from D.

Subsequent simplifications can be achieved owing to the transformation properties of the matrix elements

$$M_{nlm}^{n'l'm'}(\vec{x}) \equiv \langle \phi_{nlm} \mid \hat{O}(\vec{x}) \mid \phi_{n'l'm'}\rangle$$

with respect to the rotation group. In fact, we have

$$M_{1s}^{1s}(\vec{x}) = M_0(x^2), \qquad M_{1pm}^{1s}(\vec{x}) = M(x^2) x_m^*,$$

$$M_{1pm}^{1pm'}(\vec{x}) = M_1(x^2)\delta_{mm'} + M_2(x^2) x_m^* x_{m'} \ (m, m' = -1, 0, 1) \tag{20}$$

with $\vec{x}^* = -\vec{x}$. Here x_m are the spherical components of the vector $\vec{x}$.

In their turn, the determinants d_i can be expressed in terms of the scalar functions M_0, M, M_1 and M_2 if one takes into account that the quantities d_1 and $d_2+d_3+d_4$, each separately, are invariant under rotations, i.e., they depend on b^2, c^2 and $\vec{b}\vec{c}$. This property enables us to write down,

$$d_1 = M_1^2(c^2)\{M_0(b^2)N_0(c^2) - M(b^2)M(c^2)\vec{b}\vec{c}\}, \tag{21}$$

$$\begin{aligned} d_2 + d_3 + d_4 = M_1(c^2)\{M_1(b^2)M_0(c^2) - M(b^2)M(c^2)\vec{b}\vec{c}\}+ \\ 2M_1(b^2)M_1(c^2)\{M_0(c^2)N_0(c^2) - M^2(c^2)c^2\}+ \\ +M_0(c^2)M_1^2(c^2)M_2(b^2)b^2 - M_1(c^2)M_2(b^2)\{M_2(c^2) + M(c^2)\}\{b^2c^2 - (\vec{b}\vec{c})^2\} \end{aligned} \tag{22}$$

with $N_0(x^2) = M_{1p0}^{1p0}(x\vec{e}_0)$, where $\vec{e}_0$ is the unit vector along the Z-axis.

The scalars $M(x^2)$ and $M_1(x^2)$ satisfy the relations

$$xM(x^2) = M_{1s}^{1p0}(x\vec{e}_0), \tag{23}$$

$$M_1(x^2) = \frac{1}{2}\{M_{1p,-1}^{1p,-1}(x\vec{e}_0) + M_{1p,+1}^{1p,+1}(x\vec{e}_0)\}, \tag{24}$$

from which it follows that

$$M_1(x^2) = \exp(r_0^2 y^2/4)(A_0(y) + A_2(y)), \tag{25}$$

$$x^2 M_2(x^2) = -3\exp(r_0^2 y^2/4)A_2(y) \tag{26}$$

with

$$A_\lambda(y) = \int_0^\infty j_\lambda(yr)R_{11}(r)r^2 dr,\ \ (\lambda = 0, 2)$$

where $R_{11}(r)$ is the radial part of $\phi_{11m}(\vec{r})$ and $j_\lambda(z)$ is the spherical Bessel function of z. In Eqs. (25) − (26) $x = \imath r_0 y/\sqrt{2}$.

Thus the initial cumbersome task of handling the expectation value $f(\vec{\lambda}, \vec{q})$ is reduced to calculation of the simple overlap integrals.

All these formulae can be useful when studying CM corrections of the cross sections of the elastic and quasifree electron scattering from nuclei more complicated than the ^4He nucleus. An attractive feature of similar studies is to proceed with one and the same corrected many-body WF of the nuclear g.s. in evaluating different structure functions like the DD and MD. Of certain interest might be deviations of the q-dependence of the ratio $F(q)/F_0(q)$ beyond the HO model from that which is given by the canonical Tassie-Barker factor $\exp(q^2 r_0^2/4A)$.

Acknowledgements

We would like to thank Pavel Grigorov for technical assistance.

Appendix A. Comments on simultaneous shrinking of density and momentum distributions

Remind that a commutation relation between two noncommuting operators imposes a definite constraint upon their dispersions, i.e., the mean square deviations of these quantities from the

corresponding expectation values. In fact, let us consider the two Hermitean operators $\hat{A}$ and $\hat{B}$ that meet

$$\left[\hat{A}, \hat{B}\right] = \imath \hat{C} \; , \tag{A.1}$$

where $\hat{C}$ is also an Hermitean operator. In particular, if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}_x$, then the operator $\hat{C} = \hbar$ [4].

By definition, the expectation values with respect to an arbitrary state Φ are

$$\langle \hat{A} \rangle = \langle \Phi | \hat{A} | \Phi \rangle, \quad \langle \hat{B} \rangle = \langle \Phi | \hat{B} | \Phi \rangle \; . \tag{A.2}$$

Let us introduce the deviations

$$\widehat{\Delta A} = \hat{A} - \langle \hat{A} \rangle, \quad \widehat{\Delta B} = \hat{B} - \langle \hat{B} \rangle \; . \tag{A.3}$$

Obviously they satisfy the relation

$$\left[\widehat{\Delta A}, \widehat{\Delta B}\right] = \imath \hat{C} \; . \tag{A.4}$$

From Eq.(A.4) it follows the uncertainty relation

$$\langle (\widehat{\Delta A})^2 \rangle \langle (\widehat{\Delta B})^2 \rangle \geq \frac{1}{4} \langle \hat{C} \rangle^2 \; . \tag{A.5}$$

Note also that for operator $\hat{X}$,

$$\langle (\widehat{\Delta X})^2 \rangle = \langle (\hat{X} - \langle \hat{X} \rangle)^2 \rangle = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 \; , \tag{A.6}$$

if $\langle \Phi \mid \Phi \rangle = 1$.

In the case of interest for the relative coordinates $\hat{\boldsymbol{r}}'_\alpha$ and the canonically conjugate momenta $\hat{\boldsymbol{p}}'_\alpha$ the corresponding relation looks as

$$\langle \Phi_{int} \mid (\widehat{\Delta \boldsymbol{r}'})^2 \mid \Phi_{int} \rangle \langle \Phi_{int} \mid (\widehat{\Delta \boldsymbol{p}'})^2 \mid \Phi_{int} \rangle \geq \frac{9}{4} \left(\frac{A-1}{A} \right)^2 , \tag{A.7a}$$

or

$$\langle r^2 \rangle \langle p^2 \rangle = 9 \langle x^2 \rangle \langle {p_x}^2 \rangle \geq \frac{9}{4} \left(\frac{A-1}{A} \right)^2 \tag{A.7b}$$

for any state Φ_{int} normalized to unity. Here, in accordance with Eq.(A.6),

$$\langle \Phi_{int} \mid (\widehat{\Delta \boldsymbol{r}'})^2 \mid \Phi_{int} \rangle = \langle \Phi_{int} \mid (\hat{\boldsymbol{r}}'_1)^2 \mid \Phi_{int} \rangle \equiv \langle \boldsymbol{r}^2 \rangle = \int r^2 \rho(r) \mathrm{d}\boldsymbol{r} \tag{A.8}$$

and

$$\langle \Phi_{int} \mid (\widehat{\Delta \boldsymbol{p}'})^2 \mid \Phi_{int} \rangle = \langle \Phi_{int} \mid (\hat{\boldsymbol{p}}'_1)^2 \mid \Phi_{int} \rangle \equiv \langle \boldsymbol{p}^2 \rangle = \int p^2 \eta(p) \mathrm{d}\boldsymbol{p} \; , \tag{A.9}$$

[4] Recall that in our system of units $\hbar = 1$

since $\langle\hat{\boldsymbol{r'}}\rangle = \langle\hat{\boldsymbol{p'}}\rangle = 0$.

Thus, the general result (A.5) leads to the condition (A.7b) for the pair $\hat{\boldsymbol{r_1}}'$ and $\hat{\boldsymbol{p_1}}'$ that obeys the commutation rules,

$$\left[(\hat{\boldsymbol{r}}'_1)_j, (\hat{\boldsymbol{p}}'_1)_k\right] = \imath\delta_{jk}\frac{A-1}{A} \quad (j,k = 1,2,3). \tag{A.10}$$

Eqs.(A.10) are different from

$$[(\hat{\boldsymbol{r}}_1)_j, (\hat{\boldsymbol{p}}_1)_k] = \imath\delta_{jk} \quad (j,k = 1,2,3). \tag{A.11}$$

By the way, this fact means that the transformation $\hat{\boldsymbol{r}}_\alpha \to \hat{\boldsymbol{r}}'_\alpha$ and $\hat{\boldsymbol{p}}_\alpha \to \hat{\boldsymbol{p}}'_\alpha\,(\alpha = 1,\ldots,A)$ is nonunitary, i.e., it cannot be performed via a unitary operator $\hat{U}$ ($\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = 1$), viz., putting $\hat{U}^\dagger\hat{\boldsymbol{r}}_\alpha\hat{U} = \hat{\boldsymbol{r}}'_\alpha$ and $\hat{U}^\dagger\hat{\boldsymbol{p}}_\alpha\hat{U} = \hat{\boldsymbol{p}}'_\alpha$. Under the latter the commutation relations would not change.

It follows from (A.5) and (A.11) that

$$\langle r^2\rangle_0\langle p^2\rangle_0 \geq \frac{9}{4} \tag{A.12}$$

for the expectation values

$$\langle r^2\rangle_0 \equiv \langle\Phi \mid \hat{\boldsymbol{r}}_1^2 \mid \Phi\rangle = \int r^2\rho_0(r)\mathrm{d}\boldsymbol{r} \tag{A.13}$$

and

$$\langle p^2\rangle_0 \equiv \langle\Phi \mid \hat{\boldsymbol{p}}_1^2 \mid \Phi\rangle = \int p^2\eta_0(p)\mathrm{d}\boldsymbol{p} \tag{A.14}$$

in the shell model ground state Φ (in general, any state Φ) such that $\langle\Phi \mid \Phi\rangle = 1$.

So, we have relation (A.7b) for the mean square radius and mean square momentum in an exact translationally invariant state versus relation (A.12) obtained for similar quantities within an approximate treatment of the system (say, with the wavefunction which has a violated symmetry under space translations). In comparison with Eq.(A.12), the r.h.s. of Eq.(A.7b) is modified by the factor $\left(\frac{A-1}{A}\right)^2 \leq 1$. This modification is closely connected with nonunitarity of the transformation from the usual coordinates and canonically conjugate momenta to their relative counterparts.

Thus the simultaneous shrinking of the density and momentum distributions, shown in the paper within the HOM, is consistent with the model independent uncertainty relations (A.7).

Appendix B. Comments on derivation of Eq.(18)

One has to deal with the expectation values of type

$$A_\Phi = \langle\Phi \mid \hat{O}_1(\vec{b})\hat{O}_2(\vec{c})\ldots\hat{O}_A(\vec{c}) \mid \Phi\rangle, \tag{B.1}$$

$$\hat{O}_\gamma(\vec{x}) = \exp(-\vec{x}^*\hat{\vec{a}}_\gamma^\dagger)\exp(\vec{x}\hat{\vec{a}}_\gamma) \equiv \hat{E}_\gamma^\dagger(-\vec{x})\hat{E}_\gamma(\vec{x}), (\gamma = 1, \ldots, A) \tag{B.2}$$

where, for instance, $\vec{b} = \vec{c} + \vec{s}$. One can write

$$A_\Phi = \langle \Phi \mid \hat{E}_1^\dagger(-\vec{c})\hat{E}_2^\dagger(-\vec{c})\ldots\hat{E}_A^\dagger(-\vec{c})\hat{E}_1^\dagger(-\vec{s})\hat{E}_1(\vec{s})\hat{E}_1(\vec{c})\hat{E}_2(\vec{c})\ldots\hat{E}_A(\vec{c}) \mid \Phi\rangle. \tag{B.3}$$

We have used the properties $E_1(\vec{c}+\vec{s}) = E_1(\vec{c})E_1(\vec{s})$ and $[E_\alpha(\vec{x}), E_\beta(\vec{y})] = 0$ for any vectors $\vec{x}$ and $\vec{y}$.

If $\mid \Phi\rangle$ is a Slater determinant, i.e.,

$$\mid \Phi\rangle = \mid Det\rangle = \sqrt{A!}\hat{\Omega} \mid \phi_1(1)\rangle \cdots \mid \phi_A(A)\rangle \tag{B.4}$$

with the antisymmetrization operator

$$\hat{\Omega} = (A!)^{-1}\sum_P \varepsilon_p \hat{P} \tag{B.5}$$

which has the property

$$\hat{\Omega}^2 = \hat{\Omega}, \tag{B.6}$$

then

$$A_\Phi = \langle Det(-\vec{c}) \mid \hat{E}_1(-\vec{s})\hat{E}_1(\vec{s}) \mid Det(\vec{c})\rangle \tag{B.7}$$

with

$$\mid Det(\vec{c})\rangle = \hat{E}_1(\vec{c})\ldots\hat{E}_A(\vec{c}) \mid Det\rangle = \sqrt{A!}\hat{\Omega}\hat{E}_1(\vec{c}) \mid \phi_1(1)\rangle \ldots \hat{E}_A(\vec{c}) \mid \phi_A(A)\rangle \tag{B.8}$$

Furthermore, using the permutation symmetry of the determinants involved, viz.,

$$\hat{P} \mid Det(-\vec{c})\rangle = \varepsilon_p \mid Det(-\vec{c})\rangle \tag{B.9a}$$

$$\hat{P} \mid Det(\vec{c})\rangle = \varepsilon_p \mid Det(\vec{c})\rangle, \tag{B.9b}$$

it is easily seen that

$$\begin{aligned} A_\Phi &= \langle Det(-\vec{c}) \mid \hat{E}_1(-\vec{s})\hat{E}_1(\vec{s}) \mid Det(\vec{c})\rangle \\ &= \langle Det(-\vec{c}) \mid \hat{E}_2(-\vec{s})\hat{E}_2(\vec{s}) \mid Det(\vec{c})\rangle = \ldots \\ &= \frac{1}{A}\langle Det(-\vec{c}) \mid \sum_{\alpha=1}^{A}\hat{E}_\alpha(-\vec{s})\hat{E}_\alpha(\vec{s}) \mid Det(\vec{c})\rangle \end{aligned}$$

Now, taking into account Eq.$(B.6)$ and the relation

$$[\hat{\Omega}, \sum_{\alpha=1}^{A}\hat{B}_\alpha] = 0$$

for any A operators $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_A$, we find

$$A_\Phi = \frac{1}{A}\langle\psi_1'(1) \mid \langle\psi_2'(2) \mid \ldots \langle\psi_A'(A) \mid \sum_{\alpha=1}^{A} E_\alpha^\dagger(-\vec{s})E_\alpha(\vec{s}) \sum_p \varepsilon_p \hat{P} \mid \psi_1(1)\rangle \mid \psi_2(2)\rangle \cdots \mid \psi_A(A)\rangle \tag{B.10}$$

where we have introduced the two sets $\{\psi\}$ and $\{\psi'\}$ of new orbitals

$$\mid \psi(\alpha)\rangle = \hat{E}_\alpha(\vec{c}) \mid \phi(\alpha)\rangle \tag{B.11a}$$

and

$$\mid \psi'(\alpha)\rangle = \hat{E}_\alpha(-\vec{c}) \mid \phi(\alpha)\rangle \quad (\alpha = 1, \ldots, A). \tag{B.11b}$$

Expression (B.10) explicitly reads

$$\begin{aligned} A_\Phi = \frac{1}{A}\,[\,\sum_P \varepsilon_p \langle\psi_1'(1) \mid \hat{E}^\dagger(-\vec{s})\hat{E}(\vec{s}) \mid \psi_{p_1}(1)\rangle\langle\psi_2' \mid \psi_{p_2}\rangle \ldots \langle\psi_A' \mid \psi_{P_A}\rangle + \\ + \sum_P \varepsilon_p \langle\psi_1' \mid \psi_{P_1}\rangle\langle\psi_2'(2) \mid \hat{E}^\dagger(-\vec{s})\hat{E}(\vec{s}) \mid \psi_{p_2}(2)\rangle \ldots \langle\psi_A' \mid \psi_{P_A}\rangle + \\ + \sum_P \varepsilon_p \langle\psi_1' \mid \psi_{P_1}\rangle\langle\psi_1' \mid \psi_{P_1}\rangle \ldots \langle\psi_A'(A) \mid \hat{E}^\dagger(-\vec{s})\hat{E}(\vec{s}) \mid \psi_{p_A}(A)\rangle\,]. \end{aligned}$$

The latter is equivalent to Eq.(18).

Appendix C. Evaluation of determinants in Eq.(18)

In order to simplify evaluation of the $(1s)^4(1p)^{12}$ configuration determinants involved in the r.h.s. of Eq.(18), let us consider a sparse $\mathbf{nm} \times \mathbf{nm}$ matrix

$$Z(nm \times nm) = \begin{bmatrix} Z_{11}(m \times m) & Z_{12}(m \times m) & \ldots & Z_{1n}(m \times m) \\ Z_{21}(m \times m) & Z_{22}(m \times m) & \ldots & Z_{2n}(m \times m) \\ \ldots & \ldots & & \ldots \\ Z_{n1}(m \times m) & Z_{n2}(m \times m) & \ldots & Z_{nn}(m \times m) \end{bmatrix} \tag{C.1}$$

that consists of $\mathbf{n}^2$ diagonal $\mathbf{m}\times\mathbf{m}$ block matrices

$$Z_{ik}(m \times m) \equiv \mathrm{diag}[Z_{ik}^{(1)}, Z_{ik}^{(2)}, \ldots, Z_{ik}^{(m)}] \tag{C.2a}$$

or

$$Z_{ik}(m \times m) = \begin{bmatrix} \bullet & 0 & \ldots & 0 \\ 0 & \bullet & \ldots & 0 \\ \ldots & \ldots & & \ldots \\ 0 & 0 & \ldots & \bullet \end{bmatrix} \tag{C.2b}$$

$$(i, k = 1, 2, \ldots, n).$$

In the representation (C.2b) the diagonal elements $Z_{ik}^{(l)}$ (l=1,2,...,m) which are, in general, arbitrary and different from one another, are marked by the symbol $\bullet$. One can show that $det\ Z = \| Z(nm \times nm) \|$ is equal to the product

$$det\ Z = \prod_{i=1}^{m} det\ Y_i \tag{C.3}$$

of the **m** determinants of the **n**×**n** matrices Y_i $(i = 1, \ldots, m)$. Each of Y_i is composed of the $\mathbf{n}^2$ diagonal elements of the matrices $Z_{ik}(m \times m)$ (namely, elements $\bullet$), which one encounters when passing in the clockwise (or counter-clockwise) direction the corresponding spiral-like contour as displayed in Fig. 2.

One should note that these **m** contours are not closed (i.e., they are open), and the i-th contour begins at the element $Z_{11}^{(i)}$ on the diagonal of the block Z_{11}.

The relationship (C.3) can be proved using the simple properties of determinants. As an illustration, let us demonstrate relation (C.3) for $m = 2$ and $n = 3$:

$$\begin{Vmatrix} a_1 & 0 & b_1 & 0 & e_1 & 0 \\ 0 & a_2 & 0 & b_2 & 0 & e_2 \\ c_1 & 0 & d_1 & 0 & f_1 & 0 \\ 0 & c_2 & 0 & d_2 & 0 & f_2 \\ g_1 & 0 & h_1 & 0 & j_1 & 0 \\ 0 & g_2 & 0 & h_2 & 0 & j_2 \end{Vmatrix} = - \begin{Vmatrix} a_1 & b_1 & 0 & 0 & e_1 & 0 \\ 0 & 0 & a_2 & b_2 & 0 & e_2 \\ c_1 & d_1 & 0 & 0 & f_1 & 0 \\ 0 & 0 & c_2 & d_2 & 0 & f_2 \\ g_1 & h_1 & 0 & 0 & j_1 & 0 \\ 0 & 0 & g_2 & h_2 & 0 & j_2 \end{Vmatrix} =$$

$$= \begin{Vmatrix} a_1 & b_1 & 0 & 0 & e_1 & 0 \\ c_1 & d_1 & 0 & 0 & f_1 & 0 \\ 0 & 0 & a_2 & b_2 & 0 & e_2 \\ 0 & 0 & c_2 & d_2 & 0 & f_2 \\ g_1 & h_1 & 0 & 0 & j_1 & 0 \\ 0 & 0 & g_2 & h_2 & 0 & j_2 \end{Vmatrix} = - \begin{Vmatrix} a_1 & b_1 & e_1 & 0 & 0 & 0 \\ c_1 & d_1 & f_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & a_2 & e_2 \\ 0 & 0 & 0 & d_2 & c_2 & f_2 \\ g_1 & h_1 & j_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & g_2 & j_2 \end{Vmatrix} =$$

$$= - \begin{Vmatrix} a_1 & b_1 & e_1 & 0 & 0 & 0 \\ c_1 & d_1 & f_1 & 0 & 0 & 0 \\ g_1 & h_1 & j_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & a_2 & e_2 \\ 0 & 0 & 0 & d_2 & c_2 & f_2 \\ 0 & 0 & 0 & h_2 & g_2 & j_2 \end{Vmatrix} = \begin{Vmatrix} a_1 & b_1 & e_1 & 0 & 0 & 0 \\ c_1 & d_1 & f_1 & 0 & 0 & 0 \\ g_1 & h_1 & j_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & b_2 & e_2 \\ 0 & 0 & 0 & c_2 & d_2 & f_2 \\ 0 & 0 & 0 & g_2 & h_2 & j_2 \end{Vmatrix}.$$

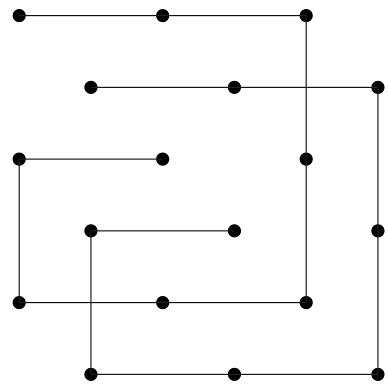

Figure 2: The relevant contours in the case with $m = 2$ and $n = 3$.

Therefore

$$\begin{Vmatrix} a_1 & 0 & b_1 & 0 & e_1 & 0 \\ 0 & a_2 & 0 & b_2 & 0 & e_2 \\ c_1 & 0 & d_1 & 0 & f_1 & 0 \\ 0 & c_2 & 0 & d_2 & 0 & f_2 \\ g_1 & 0 & h_1 & 0 & j_1 & 0 \\ 0 & g_2 & 0 & h_2 & 0 & j_2 \end{Vmatrix} = \begin{Vmatrix} a_1 & b_1 & e_1 \\ c_1 & d_1 & f_1 \\ g_1 & h_1 & j_1 \end{Vmatrix} \cdot \begin{Vmatrix} a_2 & b_2 & e_2 \\ c_2 & d_2 & f_2 \\ g_2 & h_2 & j_2 \end{Vmatrix}.$$

The aforementioned prescription yields the same result but much quicker.

In the case of interest one has to deal with the matrices of the type

$$M = \begin{bmatrix} M_{ss} & M_{s+} & M_{s0} & M_{s-} \\ M_{+s} & M_{++} & M_{+0} & M_{+-} \\ M_{0s} & M_{0+} & M_{00} & M_{0-} \\ M_{-s} & M_{-+} & M_{-0} & M_{--} \end{bmatrix},$$

where each of the sixteen 4×4 blocks is diagonal, e.g.,

$$M_{ss} = \mathrm{diag}[M_{ss}^1, M_{ss}^2, M_{ss}^3, M_{ss}^4]$$

with the four equivalent dispositions:

$$\mathrm{diag}[M_{1s}^{1s}(\vec{b}), M_{1s}^{1s}(\vec{c}), M_{1s}^{1s}(\vec{c}), M_{1s}^{1s}(\vec{c})],$$

$$\mathrm{diag}[M_{1s}^{1s}(\vec{c}), M_{1s}^{1s}(\vec{b}), M_{1s}^{1s}(\vec{c}), M_{1s}^{1s}(\vec{c})],$$

$$\mathrm{diag}[M_{1s}^{1s}(\vec{c}), M_{1s}^{1s}(\vec{c}), M_{1s}^{1s}(\vec{b}), M_{1s}^{1s}(\vec{c})],$$

$$\mathrm{diag}[M_{1s}^{1s}(\vec{c}), M_{1s}^{1s}(\vec{c}), M_{1s}^{1s}(\vec{c}), M_{1s}^{1s}(\vec{b})].$$

Taking into account this equivalence we get expression (19) which is much simpler compared to Eq.(18).

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